NYSFEA Essay Contest

For the second time N.Y.S.F.E.A. has held its annual essay contest. This has been made possible through the good grace of theoffice of Dr. Linda Bemier, Dean of Education at SUNY Binghamton, who has privided funds for the awards. The three eassays chosen by the N.Y.S.F.E.A. panel (Sue Books, Eduardo Duarte and Anthony Roda) were submitted by Sarah Dodge of Caledonia Mumford School, first plac; Jenifer L. Brown, Lead Teacher of Pre-school Age Children at the Early childhood Research Center, SUNY Buffalo, second place; and Jennifer A. Stoutenburg of Jefferson Avenue Elementary School, Fairport, New York, third place. The essays were presented at the association's annual meeting held at SUNY Oswego (April 9-10, 1999). The following is the first place essay, "Philosophical Basis for the Use of Emirical Methods in Teaching Mathematices" by Sarah Dodge.

THE PHILOSOPHICAL BASIS FOR THE USE OF EMPIRICAL METHODS IN TEACHING MATHEMATICS

Sarah Dodge

The Philosophical Basis for the Use of Empirical Methods in Teaching Mathematics

Educators use many different methods in teaching mathematics. Many rely on the lecture method and the extensive use of practice in using abstract algorithms to reinforce espoused theory. Others include more empirical methods, including the use of hands-on and discovery activities, in order to discern the mathematical ideas in the world around us. From this, the question arises of how it is that we are justified in using empirical methods in the teaching of mathematics when mathematics is essentially abstract and consists of eternal truths and ultimate forms?

Opposing Philosophical Views in Mathematics Education

Rationalism

The rationalist philosopher believes that ideas are the only true reality and that knowledge is discovered, not created. Plato (427-347 BC), one of the most influential of the rationalist philosophers, believed that the mind is superior to the body (Ediger, 1996). Plato believed that one finds perfection and eternal, unchanging truth in the Forms (Ediger, 1996). The Forms are separate from mind and body and exist independently from human existence, but can be discovered by the human mind through reasoning and dialectic inquiry (Ediger, 1996). The aim of education, for the rationalist, is this discovery of eternal, unchanging truths.

Plato believed that in the Forms, a perfect something exists, such as number systems and geometrical figures (Ediger, 1996). Mathematics is neither physical nor mental. Mathematics is independent of our wishes. (Hersh, 1997) According to Plato and his theory of the divided line, mathematics is part of the intelligible realm of being rather than the visible world of becoming (Jowett, 1998).

Such credence in the nature of mathematics implies the use of an idea centered mathematics curriculum. Students would need to think critically, synthesize content, and appraise the abstractions that would be acquired. They would discover, for example, that twelve times ten equals one hundred twenty in base ten regardless of human awareness of this fact. (Ediger, 1996) Students would see that particular, concrete instances in our world secure some of the abstract ideas that are discovered (Ediger, 1996). It is in this sense that the rationalist would find any value in experience. Concrete learning opportunities would be used only to assist and guide learners to understand the abstract, real part of mathematics (Ediger, 1996).

The discovery of mathematical ideas not found to exist in a concrete manner in our world suggests that mathematics is the logically connected study of abstract systems (Tymoczko, 1998) and further supports this type of curriculum. To the rationalist, mathematics is only empirical in that we discover it, external to ourselves (Hersh, 1997).

Empiricism

The empirical philosopher believes that all ideas originate in sense experience and that knowledge has a universal character that only comes to humans through sensory experience (Ediger, 1996). Ideas are created from experience. John Locke (1632-1704) distrusted abstract ideas. Locke states, “Let us suppose the mind to be, as we say, white paper, void of all characters, without any ideas: -How comes it to be furnished? Whence comes it by that vast store which the busy and boundless fancy of man has painted on it with an almost endless variety? Whence has it all the materials of reason and knowledge? To this I answer, in one word, from experience” (Fraser, from Locke, 1959, 121-122). Locke linked mathematical truths to moral truths. He believed that both of these seemingly abstract conceptions are simply idealized from the experiences of people (Hersh, 1997).

The metaphysics and epistemology of the empiricist supports the use of empirical methods in the creation of knowledge. Real world situations, measurable objectives, the gathering of data, precision, and exactness would all be a part of the mathematics curriculum. (Ediger, 1996).

The beliefs of some mathematicians, such as Hilary Putnam, that math is a human creation based on the physical world supports this type of curriculum. To the empiricist, abstract mathematical ideas are derived from physical possibility. (Tymoczko, 1998)

Constructivism

The Constructivist believes that truth is made, not discovered. Constructivists believe that one makes and remakes reality. The goal of education in Constructivism is the growth of active learners through the construction and reorganization of cognitive structures. This goal, also, extracts the social aspects involved in learning. (Wheatley, 1991)

Constructivism draws heavily on Piaget's work. Jean Piaget (1896-1980) preferred to be referred to as a genetic epistemologist. His theories emphasized thought process and the continuous interaction between the child and the world around him/her as being essential to cognitive development. (Duffy, 1996) Piaget believed that seeking a balance between cognitive structures and environmental demands is what drives changes in cognition.

One of Jean Piaget's greatest contributions to educational theory was his developmental stage theory. This theory was based on Piaget's idea that there is an innate biological maturation in which learners develop cognitive structures to explain and order life experiences (Piaget, 1967). According to Piaget, the development of the learner occurs in four stages, although there is no guarantee that everyone will complete all four stages. The first stage is the sensorimotor stage in infants. At this stage, knowledge is gained solely from sensory experience. The stage of preoperational thought occurs roughly between the ages of two and seven. At this stage, children begin to use symbolic representation and know that objects exist even when not experienced, although judgments are still based on appearance rather than logic. The third stage of operational thought begins around eight years of age. Thought becomes increasingly more logical and comprehension of reversible action begins. By age twelve, formal operational thought has settled in. Children are now able to apply logical thinking to theory. (Duffy, 1996)

Based on Piaget's theory, Constructivists believe that what is expected from students should be developmentally appropriate. Piaget proposed that the concepts of counting and of the natural numbers are acquired through physical activity (Hersh, 1997). Piaget believed that active participation, the active process of discovery learning, a hands-on approach in math for those who have not attained the fourth stage, yet, and interaction between the teacher and the student should all be methods used in the learning process (Duffy, 1996). For the Constructivist, the students should construct his/her own mathematical knowledge and understanding through such methods as group work, social discussion, and hands-on experiences (Watson, 1995 and Wheatley, 1991).

Pragmatism

The Pragmatist philosophy encourages the use of processes that work best to help achieve desirable ends. Pragmatists believe that there are not universals or absolute truths. They believe that true knowledge is tested in experience. (Ediger, 1996).

John Dewey (1859-1952) was an important proponent of Pragmatism. Dewey believed that every study has a form and a content side of which the form is abstract in nature and the content is concrete. He believed that the abstract should be taught with reference to its use in social life. With reference to mathematics, Dewey felt that, at the introductory stage, the concept should relate to real ends and values then, eventually, the method may become a study of its own. He believed that mathematics should not be treated as an end in itself, but as a means for accomplishing an end. (Archambault, 1974)

Pragmatists believe in using educational methods that are social and child-centered where the child learns through experience and the focus is on what works best for the individual. Curriculum should focus on practical solutions to practical problems involving hypotheses, testing in life like situations, forming conclusions, and solving problems in current society. (Ediger, 1996) Learning should be active (Archambault, 1974).

Humanism

Many twentieth century philosophies focus on humanity and its relation to the nature of reality and the attainment of knowledge. Humanism is, in this way, related to Pragmatism and its offshoots, Reconstructionism and Existentialism. Humanists believe that one must consider that reality is not only internal or external in order to understand where mathematics fits in (Brockman and Hersh, 1998). Mathematics is neither physical nor mental, but social, which makes it both internal and external (Brockman and Hersh, 1998). Humanists believe that mathematics is a human activity and a human creation. It fits into the universal part of human culture. (Hersh, 1997 and Brockman and Hersh, 1998) Mathematics is internal to humanity as a whole and external to the individual (Brockman and Hersh, 1998).

Since mathematics is both physical and mental, it can only be studied in social-cultural- historic terms (Hersh, 1997). Mathematics consists of socially held concepts, thus, interaction and communication are an essential part of mathematics education (Brockman and Hersh, 1998). Empirical evidence and numerical experimentation are necessary in the decision of what to believe in mathematics and what may be mistakes in need of correction (Hersh, 1997). Pedagogical method would stress problem solving and other real experiences that are important to the individual student. The personal needs of each learner would be met. (Ediger, 1996) Each person involved would be both a learner and a teacher. One of the most important outcomes would be that the student learns, through social interaction, that mathematics is both discovered and created. Reuben Hersh gives the example that “Counting numbers are discovered. Pure numbers were invented.” (Hersh, 1997, 75) Students would realize the transformation of the potential to the actual.

Kant’s Attempted Unification

In summary, so far, the philosophies of rationalism, empiricism, Constructivism, Pragmatism, and Humanism support some type of empirical method used in the teaching of mathematics, although for different reasons. The discrepancies between these philosophies are unsettling and unsatisfactory in our search for the justification of the use of empirical methods in mathematics education. A comprehensive philosophy unifying the major divarications may provide a more convincing, stronger justification for these methods. Immanuel Kant (1724-1804) sought to unite rationalism and empiricism (Morse, 1998). Kant believed that reason and experience both contribute to knowledge. He thought that concepts without perceptions are empty and perceptions without concepts are blind (Kant's Metaphysics, 1998). Thought processes organize the data from experience and data from experience gives new insights.

Kant describes analytic a priori ideas and synthetic a posteriori ideas. Ideas that follow a logical thought process are analytic. Ideas that use empirical investigation for confirmation are synthetic. A priori knowledge is constructed by the mind. A posteriori knowledge is the result of experience. Kant states that common knowledge and scientific knowledge are a posteriori because this knowledge comes from the material world. Mathematical knowledge is a priori because it is independent of the material world and timeless. Kant further makes the distinction between analytic a priori, which is knowledge by analytic analysis, and synthetic a priori, which is not knowledge by logical truisms, but comes from the intuitions. (Hersh, 1997)

In *Prolegomena to Any Future Metaphysics*, Kant states that synthetic a priori statements are only permissible in mathematics (The Internet Encyclopedia of Philosophy, 1997). To Kant, pure mathematics is synthetic a priori (Hersh, 1997 and McFarlane, 1995). This belief is established on the basis of the universal human intuitions. Kant believes that the only innate knowledge that humans are born with are the intuitions of space and time. Kant states that arithmetic is the systematization of the intuition of time and that the intuition of space is systematized in geometry and that these are true for everyone, independent of experience (Hersh, 1997). It would follow that the validity of mathematical knowledge rests on the fact that it is based on the a priori forms of our sensibility that condition the possibility of experience (McFarlane, 1995 and Tymoczko, 1998). Based on such beliefs, abstract and empirical methods would be used together in the teaching of mathematics. Empirical methods would be used to model the abstract concepts, but also to lead to the development of new abstract inquiry. (Hersh, 1997) The form of mathematics is given to us a priori, but the content is given to us in sensation (McFarlane, 1995).

The difficulty with Kant's philosophy lies in the fact that Euclidean geometry was the only geometry that was well established at that time (Hersh, 1997 and McFarlane, 1995). Many mathematicians find this to be a source of weakness in Kant's theories. They argue that Euclidean geometry can not be completely predetermined by our pure intuition of space alone because of the discovery of Non-Euclidean geometries (Hersh, 1997 and McFarlane, 1995). While Euclidean geometry explains the visual world around us, Non-Euclidean geometry is a better model for space beyond Earth, about which little was known in Kant's time. The two geometries are at odds because Euclidean geometry assumes the postulate that every line has exactly one parallel line through a particular point not on the original line and the Non-Euclidean geometries, such as Hyperbolic and Elliptic, assume that there are more than one parallel lines through a point or no parallels at all. The mathematicians argue that since these geometries can not coexist, it is impossible to have an intuition of space that implies geometry (Hersh, 1997).

Other mathematicians argue that the intuitive character of mathematics means that it is limited to objects that can be constructed and that intuition limits the broader region of mathematical existence. They claim that Kant knew that other geometries may be possible, but that they cannot be constructed as an image or picture even if physical existence is achieved. These mathematicians claim that Kant used Euclidean geometry because it was not a problematic premise in a social sense. They support this belief with Kant's statement that the ground for the three-fold dimension of space is unknown and arbitrary and is not logically necessary. To them, Euclid's geometrical system is a transcendental abstraction from actual experience and the physical world does not necessarily conform to the way we must view it. In light of this perspective, the question arises as to whether or not some other geometry, though not picturable to our sensibility, might conform more closely to the way objects are actually structured in empirical space. (Kant on Euclid, 1998)

Whitehead's Solution

Alfred North Whitehead (1861-1947) sought to reconcile some of the aspects of rationalism and empiricism. Process is central to Whitehead's philosophy because he believed that reality is process (Stanford Encyclopedia of Philosophy, 1997). One encounters in this process occasions, which are objects, prehensions, which are the relational processes between the person and the objects experienced, and nexus, which is the extended time sequence when occasions and prehensions coalesce in ongoing existence (Whitehead, 1929). Whitehead believed that all objects should be understood as fields having both temporal and spatial extensions and that each object can be understood as a series of events and processes (Stanford Encyclopedia of Philosophy, 1997). Whitehead did not believe in the separation of the mind into a realm of its own. He thought that mental activity had to be viewed in the context of experience. But, Whitehead also believed that "The understanding of actuality requires reference to ideality." (Whitehead, 1953, 158). The empirical and the abstract are interdependent. What Whitehead refers to as eternal objects, similar to forms and universals, are abstract in that they are comprehensible without reference to a particular occasion of experience (Whitehead, 1953). Yet, a particular individual eternal object is connected to actual empirical occasions of objects, has a general relationship to other eternal objects, and still has a general character of its own because of the realm of possibility (Whitehead, 1953). In this way, what is real is both mental and physical and the acquisition of knowledge requires both the abstract and the empirical.

Whitehead thought that the important things to be learned are ideas and that these ideas should be useful and connected with the experiences of the students. He believed that one learns best from the material world in which one lives. Whitehead cautions against the teaching of inert ideas that are received into the mind without being tested and used. (Whitehead, 1949) He states that inert ideas are useless and harmful and suggests that, to guard against the resulting "mental dryrot", students should not be taught too many subjects and that what is taught should be taught thoroughly (Whitehead, 1949, 2). Too many small parts of a large number of subjects results in the passive reception of disconnected ideas, and passivity and inertness are a hazard in education (Whitehead, 1949).

Whitehead thought that the child should enjoy the experience of discovery. The knowledge that the child gains should be applicable to actual life in the present and the child should feel that he/she owns the knowledge. Education needs to be useful. Ideas should relate to the students' sense perceptions, feelings, hopes, desires, and mental activities, adjusting thought to thought. Whitehead believed that there should be interdisciplinary connections between subjects because there is only one true subject matter and that is Life. (Whitehead, 1949)

Whitehead held that it is important to study the applications of a theoretical subject and that theoretical exposition should be short, strict, and rigid. He believed that the essential part of an idea is proving its truth either by experiment or logic. Students need to appreciate the importance of ideas and Whitehead believed that an idea is best appreciated through its use. Although, the use of an idea should not just include neat little experiments to demonstrate or prove isolated propositions. The use of interrelated truths is important. He, also, believed that one way of learning will not suit all learners and that specialization is needed at the more advanced stages according to students' interests. (Whitehead, 1949)

Whitehead believed that education is the acquisition of the art of the utilization of knowledge and stated, "What education has to impart is an intimate sense for the structure of ideas, together with a particular body of knowledge which has peculiar reference to the life of the being possessing it.” (Whitehead, 1949, 23).

Whitehead believes that pure mathematics is a realm of complete and absolute abstraction. He believes that mathematical ideas are discovered. (Whitehead, 1953) Yet, Whitehead believes that the science of mathematics is concerned with the investigations of patterns of connectedness, which not only applies to the abstract, but, also, the empirical (Whitehead, 1955). But, there are mathematical science connections between things that are extremely unobvious and we must depend on some prior, more abstract, mathematical knowledge to further our mathematical knowledge because our notions have progressed much further than those derived by sense perception (Whitehead, 1953). Once again, though, we return to the empirical nature of mathematics and the importance of the analysis of concrete fact because, as theory becomes more complex through the ages, new applications arise (Whitehead, 1953). Whitehead states, “The generality of mathematics is the most complete generality consistent with the community of occasions which constitute our metaphysical situation.” (Whitehead, 1953, 25).

Based on his beliefs about the nature of mathematics, Whitehead thinks that educators need to strengthen habits of concrete appreciation of individual facts in their interplay within mathematics and between mathematics and the rest of the world (Whitehead, 1953). Whitehead thinks that the reason that mathematics is a delight, but, also, an obstruction in education is “...the boundless wealth of deductions from the interplay of general theorems, their complication, their apparent remoteness from the ideas from which the argument started, the variety of methods, and their purely abstract character which brings, as its gift, eternal truth.” (Whitehead, 1949, 84). He believes that, in education, mathematics must be subjected to selection and adaptation, dealing directly and simply with a few general ideas of far reaching importance. Educators should eliminate the abstrusity of mathematics and focus on the relations of numbers, the relations of quantities, the relations of space, and the interconnectedness of these ideas. (Whitehead, 1949) The material taught must be relevant to modern thought. The student should acquire familiarity with abstract thought and realize how it applies to concrete circumstances. The degree of abstraction and concreteness should be age appropriate, according to the stages mentioned previously. Students need numerous examples, both abstract and concrete.

Whitehead believes that the use of both induction and deduction are necessary. (Whitehead, 1953) He states, "There is a tradition of opposition between adherents of induction and of deduction. In my view it would be just as sensible for the two ends of a worm to quarrel." (Quotations by Alfred Whitehead, 1998). Whitehead believes that, initially, one should go from abstract and concrete particulars to general ideas, avoiding the pointless accumulation of details, so that the details illustrate the main ideas. Once the general is grasped, it can be applied to more particulars. (Whitehead,1953) Whitehead states that bookwork is good if used properly, but the best method "...should satisfy the itch of youth to be doing something." (Whitehead, 1953, 198).

Alfred North Whitehead's general philosophy, view of the aims of education, and beliefs about the nature of mathematics and mathematics education based on philosophy overwhelmingly support the use of both abstract methods and empirical methods in the teaching of mathematics.

Applications of Whitehead's Philosophy

The impact of Whitehead's philosophy is apparent in applications in education.

The Illinois Mathematics and Science Academy is applying Whitehead's idea "...to eradicate the fatal disconnection of subjects which kills the vitality of our modern curriculum. There is only one subject-matter for education, and that is Life in all its manifestations."

(Eggebrecht, et. al., 1996, 4). The academy is reconstructing Whitehead's "one subject matter" by creating an Integrated Science program which reconnects biology, chemistry, earth and space sciences, physics, and the mathematics involved in each. Each student engages in interconnected experiences in response to his/her own inquiry as opposed to a static textbook interpretation of what should be learned. This learner-centered method of engaging students in multiple contexts enhances the transfer of knowledge from the familiar to new situations, which is an important life skill. (Eggebrecht, et. al. 1996)

Research shows that activity-oriented discovery in mathematics, an empirical method promoted by Whitehead, which involves collecting data or making observations, searching for patterns, and formulating conjectures, is very important in problem solving. Analyzing the data collected and connecting that to the prior knowledge of the individual results in theoretical conjectures created by the individual. (Kazuhiko, 1997) The analysis of concrete leads to a better understanding of the abstract. One two-year study found that student confidence, interest, and ability in solving algebraic equations were high when using hands-on methods, such as working with manipulatives (Leinenbach, 1996).

The National Council of Teachers of Mathematics, also, advocate the use of many of the same methods that Whitehead encouraged. The Council calls for teachers to consider what they know about their students and students' interests when choosing methods (NCTM, 1991). They encourage teachers of mathematics to use technology and concrete materials to explore mathematical ideas (NCTM, 1991), such as manipulatives (NCTM,1989). The Council holds that mathematics is connected with other fields (NCTM, 1991) and connections should be made between mathematical concepts themselves and life circumstances (NCTM, 1989). Educators should apply mathematics to life problems and settings that are relevant to the students (NCTM, 1989). The Council, also, encourages group work where students can ask questions, discuss ideas, learn to listen to others, learn to learn from mistakes, and summarize discoveries (NCTM, 1989).

Conclusion

Based on our consideration of the philosophies of rationalism, empiricism, Constructivism, Pragmatism, Humanism and the unification of ideas in the philosophies of Immanuel Kant and Alfred North Whitehead, it is clear that empirical methods have a place in mathematics education. I believe that Whitehead's interpretation provides the strongest justification of the use of empirical methods and the most comprehensive explanation of the interconnectedness of empirical methods with relation to other methods. Mathematics is neither physical nor mental, but has both physical and mental manifestations. The learner can discover true ideas through thought and experience, but, also, can create a personalized form of knowledge. The abstract study of mathematics is necessary, but it must build upon the concrete so that the learner can appreciate mathematics for its usefulness and, eventually, for its beauty. A child-centered, social, interdisciplinary approach to mathematics using empirical methods and abstract processes, as promoted by Whitehead, will help students develop into confident, fulfilled individuals and life long learners.

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