**The Four Color Theorem**

**Theorem (4CT)**
Every map in the plane (or on the sphere) can be colored with just four colors such that no two adjacent countries have the same color.

- First stated by Francis Guthrie in 1852.
- First proofs submitted by Kempe in 1879 and Tait in 1880.
- Both proofs were shown to be incorrect, more than ten years after they were submitted.

**Computer Proofs**

- First correct proof by Haken and Appel used computers to verify the 4CT in 1976.
- Proof uses 1,476 configurations each checked by computer for reducibility and unavoidability.
- Robertson, Sanders, Seymour, and Thomas simplified Haken and Appel’s proof in 1997.
- Proof uses 633 configurations each checked by computer for reducibility and unavoidability.
- Gonthier uses Coq proof assistant to write a formal proof of the 4CT in 2005.
- This proof follows the Robertson, Sanders, Seymour, and Thomas’s proof.
- As a formal proof, one only needs to believe the Coq software is error free.
- While these proofs are generally excepted, they are difficult to understand without considerable study.

**An Equivalent Reformation**

We would still like an alternate proof of the 4CT. One that does not rely on computers or a large number of cases to understand.

**Theorem (Tait, 1880)**
A graph is 4-colorable if and only if the edges can be colored with 3 colors such that no two edges which share an endpoint have the same color.

This can be shown by taking the 4 colors to be elements of $\mathbb{Z}_2^2$.

**Theorem (Vector Association Theorem)**
Given any two associations of the cross product $v_1 \times v_2 \times \cdots \times v_n$, there is an assignment of vectors $i$, $j$, and $k$ such that both products are equal and non-zero.

It should be surprising that what seems like a question about combinatorics is equivalent to a question about algebra.

**Theorem (Kaufman, 1990)**
The Four Color Theorem is equivalent to the Vector Association Theorem.

Since the Vector Association Theorem is about associativity, Thompson’s Group $F$ is a natural place to look for an alternate proof of the 4CT.

**Thompson’s Group $F$**

An element of Thompson’s Group $F$ is a pair $(L, R)$ of two binary trees each with the same number of leaves. Thompson’s Group $F$ is generated by the elements $x_i$, for $i \geq 0$.

$$x_0 = (\lambda, \lambda), \quad x_1 = (\lambda, \lambda), \quad x_i = (\lambda, \lambda)$$

To compute the product $fg$, you add carets to both trees of $f$ and $g$ until the center tree’s match, the product is then the new left and right trees of $f$ and $g$.

$$(\lambda, \lambda)(\lambda, \lambda) = (\lambda, \lambda)(\lambda, \lambda) = (\lambda, \lambda)$$

**Definition**
A coloring is an assignment of elements of $2^2_2$ to the leaves of $f$ such that evaluation on both the left and right trees yields the same result, and no node of either tree is given the identity color.

**Positive Elements of $F$ Are Colorable**

**Definition (positive element)**
An element $f$ of $F$ is called positive if $f = x_0^{i_0}x_1^{i_1} \cdots x_n^{i_n}$ with all $i_k \geq 0$.

If one could prove that any element of $F$ is colorable, then we would have an alternate proof of the 4CT. As progress towards that goal, it can be shown that all positive elements are colorable.

**Lemma**
Every tree coloring is equivalent to a 2 coloring of its carets.

$$(\lambda, \lambda) = (\lambda, \lambda)(\lambda, \lambda) = (\lambda, \lambda)$$

Two colored elements $f, g \in F$ are compatible if the 2 colorings of $f$’s right tree and $g$’s left tree match on their intersection.

**Lemma**
If two elements of $F$ have compatible colorings, then their product is colorable.

$$f = (\lambda, \lambda) \quad g = (\lambda, \lambda)$$

**Lemma**
Given any coloring of the generator $x_i$, there is a compatible coloring for $x_j$ so long as $j \geq i$.

**Theorem (Bowlin, 2011)**
Every positive element of $F$ is colorable.

This is an application of the above lemmas to the normal form of a positive element of $F$. 

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